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ON THE DUAL OF AN EXPONENTIAL SOLVABLE LIE GROUP

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ABSTRACT. Let G be a connected, simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$. The Kirillov mapping $\eta\colon \mathfrak{g}^*/\operatorname{Ad}^*(G) \to \hat{G}$ gives a natural parametrization of $\hat{G}$ by co-adjoint orbits and is known to be continuous. In this paper a finite partition of $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ is defined by means of an explicit construction which gives the partition a natural total ordering, such that the minimal element is open and dense. Given $\pi \in \hat{G}$, elements in the enveloping algebra of $\mathfrak{g}_c$ are constructed whose images under π are scalar and give crucial information about the associated orbit. This information is then used to show that the restriction of η to each element of the above-mentioned partition is a homeomorphism.

1. Introduction. Let G be a real, connected, simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$. By a representation of G we shall mean a strongly continuous, unitary representation of G in some Hilbert space, and we denote the dual of G by $\hat{G}$, that is, the set of unitary equivalence classes of topologically irreducible representation of G. Denote by η the natural mapping of the set $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ of co-adjoint orbits in the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ onto $\hat{G}$. When $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ is given the quotient topology and $\hat{G}$ the hull kernel topology, η is continuous. It was first conjectured by A. A. Kirillov in [**8**] and proved by I. Brown in [**3**] that if G is nilpotent, η is a homeomorphism. K. Joy in a later paper [**7**] gives a much shorter proof of Brown's Theorem using results of J. M. G. Fell pertaining to the space $S(G)$ of subgroup representation pairs (π, H), where H is a closed connected subgroup of G and π is an unitary equivalence class of representations of H. Two results on the bicontinuity of η when G is exponential are due to J. Boidol [**2**] and H. Fujiwara [**6**]. Boidol shows that η^{-1} is continuous provided that G is $*$-regular; $*$-regularity is seen to fail however even for a completely solvable group of dimension four. On the other hand, Fijuwara proves the existence of a dense open subset U of $\hat{G}$ such that $V = \eta^{-1}(U)$ is dense and such that the restriction of η to V is a homeomorphism. However, Fujiwara's result provides no explicit characterization of U. Finally, it is known that η is a homeomorphism for all G of dimension less than six. Those cases which are not $*$-regular are handled by constructing elements W in the center of the enveloping algebra $U(\mathfrak{g}_c)$, and using the fact that the mapping ϕ_W on $\hat{G}$ given by $\rho(W) = \phi_W(\rho)I$ is continuous. $\phi_W \circ \eta$ can be regarded as an $\operatorname{Ad}^*(G)$-invariant polynomial function of $\mathfrak{g}^*$, and as such provides enough information to conclude convergence of the corresponding orbits. In the general case the center of $U(\mathfrak{g}_c)$ is not large enough to yield sufficient information about η^{-1}.

Now let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, and let $\rho \in \hat{G}$ such that ρ is extended from $N = \exp(\mathfrak{n})$. A generalization of the construction mentioned above is given whereby

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elements $w_i \in U(\mathfrak{g}_c)$ are constructed such that $\{\rho(W_i)\}$ are scalar operators whose values allow one to systematically obtain $\eta^{-1}(\rho)$ from the orbit of $\rho|_N$. The Kirillov mapping has a natural generalization in the context of the space of subgroup representation pairs (ρ, H) such that $H \supset N$ and $\rho \in \hat{H}$, and a theorem regarding this mapping is proved which has as a corollary the following. There is a finite partition $\{U_\alpha\}$ of $\mathfrak{g}^*/\operatorname{Ad}^*(G)$—obtained by an explicit construction depending only on a choice of Jordan-Hölder sequence for η—on each element of which η is open.

2. Preliminaries. Let $\mathfrak{g}$ be a real, solvable Lie algebra of exponential type. For any subspace $\mathfrak{h}$ of $\mathfrak{g}$, let $\mathfrak{h}^*$ denote the dual space of $\mathfrak{h}$, and if $\mathfrak{j}$ is a subspace of $\mathfrak{g}$ such that $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{j}$ and $\lambda \in \mathfrak{j}^*$, denoted by B_λ the bilinear form defined on $\mathfrak{h}$ by $B_\lambda(X,Y) = \lambda([X,Y])$, $X, Y \in \mathfrak{h}$. For any subset $\mathfrak{s}$ of $\mathfrak{h}$, denote by $\mathfrak{s}^{\lambda,\mathfrak{h}}$ the orthogonal complement of $\mathfrak{s}$ in h with respect to B_λ. The radical $\mathfrak{h}^{\lambda,\mathfrak{h}}$ of B_λ will also be denoted by $R(\lambda, \mathfrak{h})$.

Let $\{\mathfrak{h}_n\}_{n=1}^\infty$ be a sequence of subspaces of $\mathfrak{g}$. We shall say that $\mathfrak{h}_n$ converges to a subspace $\mathfrak{h}$ (or write $\mathfrak{h}_n \to \mathfrak{h}$) if there are positive integers K and d such that for each $n > K$, there is a basis $X_1^{(n)}, X_2^{(n)}, \ldots, X_d^{(n)}$ of $\mathfrak{h}_n$ and a basis $X_1, X_2, \ldots, X_d$ of $\mathfrak{h}$ with $X_j = \lim_n X_j^{(n)}$, $1 \le j \le d$. Suppose that $\mathfrak{h}_n \to \mathfrak{h}$, and let $W_n \in \mathfrak{h}_n$, $n \ge 1$, such that for some $W \in \mathfrak{g}$, $W = \lim_n W_n$. Then $W \in h$, and it follows that if for some $\mathfrak{h}', \mathfrak{h}_n \to \mathfrak{h}'$, then $\mathfrak{h}' = \mathfrak{h}$, and if $\mathfrak{h}_n$ is a subalgebra (ideal) for infinitely many n, then $\mathfrak{h}$ is a subalgebra (ideal). Clearly every sequence $\{\mathfrak{h}_n\}$ of nontrivial subspaces of $\mathfrak{g}$ has a subsequence which converges, and it is easily seen that $\mathfrak{h}_n \to \mathfrak{h}$ if and only if every convergent subsequence of $\{\mathfrak{h}_n\}$ converges to $\mathfrak{h}$.

LEMMA 2.1. *Let $\{\mathfrak{j}_n\}_{n=1}^\infty$ be a sequence of subspaces of $\mathfrak{g}$ such that for each n, $\mathfrak{j}_n \subset \mathfrak{h}_n$, and suppose that $\mathfrak{j}_n \to \mathfrak{j}$ and $\mathfrak{h}_n \to \mathfrak{h}$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathfrak{g}^*$ such that for some $\lambda \in \mathfrak{g}^*$, $\lambda|_{[\mathfrak{g},\mathfrak{g}]} = \lim_n \lambda_n|_{[\mathfrak{g},\mathfrak{g}]}$, and $\dim_{\mathbf{R}}(\mathfrak{j}^{\lambda,\mathfrak{h}}) = \liminf_n \dim_{\mathbf{R}}(\mathfrak{j}_n^{\lambda_n,\mathfrak{h}_n})$. Then $\mathfrak{j}_n^{\lambda_n,\mathfrak{h}_n} \to \mathfrak{j}^{\lambda,\mathfrak{h}}$.*

PROOF. Let K and d be positive integers such that for each $n > K$, there is a basis $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_d^{(n)}$ of $\mathfrak{j}_n$ with $\lim_n Y_j^{(n)} = Y_j$, $1 \le j \le d$, and $\{Y_j\}$ a basis of $\mathfrak{j}$. Let $\{\mathfrak{j}_k^{\lambda_k,h_k}\}$ be any convergent subsequence, $\mathfrak{j}_k^{\lambda_k,h_k} \to \mathfrak{j}_0$, and let $W \in \mathfrak{j}_0$. Then for each k, there is $W^{(k)} \in \mathfrak{j}_k^{\lambda_k,h_k}$ such that $W = \lim_k W^{(k)}$, and we have

$$\lambda([W, Y_j]) = \lim_k \lambda_k([W^{(k)}, Y_j^{(k)}]) = 0,$$

$1 \le j \le d$. Thus $\mathfrak{j}_0 \subset \mathfrak{j}_{\lambda,\mathfrak{h}}$. On the other hand,

$$\dim_{\mathbf{R}}(\mathfrak{j}_0) \ge \liminf_n \dim_{\mathbf{R}}(\mathfrak{j}_n^{\lambda_n,\mathfrak{h}_n})$$

so that $\mathfrak{j}_0 = \mathfrak{j}^{\lambda,\mathfrak{h}}$, and hence $\mathfrak{j}_n^{\lambda_n,\mathfrak{h}_n} \to \mathfrak{j}^{\lambda,\mathfrak{h}}$. □

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. We denote by $U(\mathfrak{h})$ the enveloping algebra of $\mathfrak{h}$ and regard $U(\mathfrak{h})$ as a subalgebra of $U(\mathfrak{g})$. We denote the complexification $\mathfrak{h} \otimes_{\mathbf{R}} \mathbf{C}$ by $\mathfrak{h}_c$ and regard $U(\mathfrak{h}_c)$ as a subalgebra of $U(\mathfrak{g}_c)$.

Let G be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$, and let H be the closed, connected subgroup of G with Lie algebra $\mathfrak{h}$. Denote by η_H Kirillov mapping $\mathfrak{h}^*/\operatorname{Ad}^*(H) \to \hat{H}$, let π be a representation of H, and let $\lambda \in \mathfrak{h}^*$. We shall say that π corresponds to λ if $\pi \in \eta_H(\operatorname{Ad}^*(H)\lambda)$. If $\mathfrak{p} \in \mathfrak{h}^*$ is a polarization at λ, we occasionally use the notation $\operatorname{ind}(\lambda, \mathfrak{p})$ for the irreducible representation

$\operatorname{ind}(\chi_\lambda, P, H)$ of H induced by the character χ_λ of $P = \exp(\mathfrak{p})$ with differential $i(\lambda|_{\mathfrak{p}})$.

Now let $\lambda \in \mathfrak{g}^*$, and let $\mathfrak{m}$ be a nilpotent subalgebra of $\mathfrak{g}$.

DEFINITION 2.2. A pair $(\mathfrak{m}_1, \mathfrak{m}_0)$ of $\mathfrak{m}$-ideals such that $\mathfrak{m}_0 \subset \mathfrak{m}_1$, $\dim_{\mathbf{R}}(\mathfrak{m}_1/\mathfrak{m}_0) = 1$, $\mathfrak{m}_0 \subset R(\lambda, \mathfrak{m})$ and $\mathfrak{m}_1 \not\subset R(\lambda, \mathfrak{m})$ will be called a Kirillov pair in $\mathfrak{m}$ at λ.

Let $(\mathfrak{m}_1, \mathfrak{m}_0)$ be a Kirillov pair in $\mathfrak{m}$ at λ, and let $\mathfrak{l} = \mathfrak{m}_1^{\lambda,\mathfrak{m}}$. Then $\mathfrak{l}$ is a codimension 1 subalgebra of $\mathfrak{m}$. Let π_1 be an irreducible representation of $L = \exp(\mathfrak{l})$ corresponding to $\lambda|_{\mathfrak{l}}$, and let $X \in \mathfrak{m} \sim \mathfrak{l}$. Then the representation $\pi = \pi(\pi_1, X)$ defined in $L^2(\mathbf{R}, H(\pi_1))$ by the formula

$$(1) \quad (\pi(y \exp sX)f)(t) = \pi_1(\exp tXy \exp -tX)f(t+s) \qquad (y \in L,\ s, t \in \mathbf{R})$$

corresponds to $\lambda|_{\mathfrak{m}}$. The primary representation $\tilde{\pi}_1$ defined in $L^2(\mathbf{R}, H(\pi_1))$ by $(\tilde{\pi}_1(y)f)(t) = \pi_1(y)f(t)$, $y \in L$, can be differentiated in the space $C^\infty(\pi)$ of smooth vectors for π, that is, $C^\infty(\tilde{\pi}_1) \supset C^\infty(\pi)$. The following lemma is more or less well known, but crucial in this paper.

LEMMA 2.3. *There is an explicit construction by which, given any $W \in U(\mathfrak{l}_c)$, one obtains $\tilde{W} \in U(\mathfrak{l}_c)$ such that $\pi(\tilde{W}) = \tilde{\pi}_1(W)$.*

PROOF. Let m be a positive integer such that $\operatorname{ad} X^{m+1} \equiv 0$, and let $W \in U(\mathfrak{l}_c)$. We construct an element $W_m \in U(\mathfrak{l}_c)$ as follows. Let (by abuse of notation) t denote the operator on $C^\infty(\pi)$ defined by $\phi(t) \to t\phi(t)$. We have $\pi(W) = \sum_{j=0}^{m}(t^j/j!)\tilde{\pi}_1(\operatorname{ad} X^j W)$ so that $\pi(\operatorname{ad} X^m W) = \tilde{x}_1(\operatorname{ad} X^m W)$. Let Y be the element in $\mathfrak{m}_1 \sim \mathfrak{m}_0$ such that $\lambda(Y) = 0$, and $B_\lambda(X, Y) = 1$, so that $\pi(Y) = it$. Define $W_1 \in U(\mathfrak{l}_c)$ by

$$W_1 = W - \frac{\operatorname{ad} X^m W \cdot (-iY)^m}{m!}.$$

Then $\pi(W_1) = \sum_{j=0}^{m-1}(t^j/j!)\pi_1(\operatorname{ad} X^j W)$ and $\pi(\operatorname{ad} X^{m-1} W_1) = \tilde{\pi}_1(\operatorname{ad} X^{m-1} W)$. If $m > 1$, set

$$W_2 = W_1 - \frac{\operatorname{ad} X^{m-1} W_1 \cdot (-iY)^{m-1}}{(m-1)!}$$

and we find that $\pi(\operatorname{ad} X^{m-2} W_2) = \tilde{\pi}_1(\operatorname{ad} X^{m-2} W)$. Continue in this way until $W_m = \tilde{W}$ is obtained. Q.E.D.

3. A partition of the dual of a nilpotent Lie group. Now let us assume that $\mathfrak{g}$ is nilpotent; fix $\lambda \in \mathfrak{g}^*$. By induction on the dimension of $\mathfrak{g}$ it is easily seen that there is a sequence of subalgebras $\mathfrak{g} = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_d$ satisfying the conditions

(i) $\mathfrak{m}_d$ is a polarization at λ.

(ii) If $R(\lambda, \mathfrak{g}) \neq \mathfrak{g}$, then $d > 0$ and for each $k = 1, 2, \ldots, d-1$, there is a Kirillov pair $(\mathfrak{m}_{k1}, \mathfrak{m}_{k0})$ in $\mathfrak{m}_k$ at λ such that $\mathfrak{m}_{k+1} = \mathfrak{m}_{k1}^{\lambda,\mathfrak{m}_k}$. Thus if $R(\lambda, \mathfrak{g}) \neq \mathfrak{g}$, then $d = \frac{1}{2}\dim(\operatorname{Ad}^*(G))\lambda)$.

DEFINITION 3.1. A sequence of subalgebras satisfying conditions (i) and (ii) above will be called a Kirillov sequence for λ in $\mathfrak{g}$.

Let d be a nonnegative integer. Let us say that an operator D on $C^\infty(\mathbf{R}^d)$ ($C^\infty(\mathbf{R}^0) \equiv \mathbf{C}$) is a polynomial differential operator if there is a polynomial P in $2d$ indeterminants with complex coefficients such that

$$D = P(t_1, \ldots, t_d, \partial/\partial t_1, \ldots, \partial/\partial t_d).$$

A theorem of Kirillov (cf. [8, Theorem 7.1]) states that $\eta(\mathrm{Ad}^*(G)\lambda)$ has a realization π in a space of functions on $\mathbf{R}^d$ such that the image of $U(\mathfrak{g}_c)$ under π is the set of polynomial differential operators. In this section we shall determine when it is possible, given a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathfrak{g}^*$ such that $\lambda_n \to \lambda = \lambda_0$, to obtain a corresponding sequence $\{\pi_n\}_{n=0}^{\infty}$ of irreducible representations such that given any D as above, there is a sequence $\{W_n\}_{n=0}^{\infty}$ in $U(\mathfrak{g}_c)^{(m)}$ for some m, with $W_n \to W_0$ and $\pi_n(W_n) = D$ for each n.

Let $\mathfrak{g} = \mathfrak{g}_p \supset \mathfrak{g}_{p-1} \supset \cdots \supset \mathfrak{g}_0 = (0)$ be a Jordan-Hölder sequence for $\mathfrak{g}$. Define subsets $e(\lambda)$, $j(\lambda)$ and $i(\lambda)$ of $\{1, 2, \ldots, p\}$ as follows. Set

$$e(\lambda) = \{t | \mathfrak{g}_t + R(\lambda, \mathfrak{g}) \supsetneqq \mathfrak{g}_{t-1} + R(\lambda, \mathfrak{g})\}$$

and let $\mathfrak{p}(\lambda) = \sum_t R(\lambda, \mathfrak{g}_t)$. Define $j(\lambda) \subset e(\lambda)$ by

$$j(\lambda) = \{t | \mathfrak{g}_t + \mathfrak{p}(\lambda) \supsetneqq \mathfrak{g}_{t-1} + \mathfrak{p}(\lambda)\}$$

and let $i(\lambda) = e(\lambda) \sim j(\lambda)$. Then $\mathrm{card}(e(\lambda)) = \dim(\mathrm{Ad}^*(G)\lambda)$ and it is shown in [1] that $\mathfrak{p}(\lambda)$ is a polarization at λ, hence $\mathrm{card}(j(\lambda)) = \frac{1}{2}\mathrm{card}(e(\lambda))$. If $e(\lambda) \neq \{\phi\}$, we shall write $e(\lambda) = \{e_1 < e_2 < \cdots < e_{2d}\}$. We define a sequence of subalgebras $\mathfrak{g} = \mathfrak{g}^0(\lambda) \supset \mathfrak{g}^1(\lambda) \supset \cdots \supset \mathfrak{g}^d(\lambda)$ as follows. Setting $\mathfrak{g}^0(\lambda) = \mathfrak{g}$, assume that for some $k \geq 0$, $\mathfrak{g}^k(\lambda)$ is defined and $\mathfrak{g}^k(\lambda) \neq R(\lambda, \mathfrak{g}^k(\lambda))$. Let i_{k+1} be the smallest index such that $\mathfrak{g}_{i_{k+1}} \cap \mathfrak{g}^k(\lambda) \not\subset R(\lambda, \mathfrak{g}^k(\lambda))$ and set

$$\mathfrak{g}^{k+1}(\lambda) = \left(\mathfrak{g}_{i_{k+1}} \cap \mathfrak{g}^k(\lambda)\right)^{\lambda, \mathfrak{g}^k(\lambda)}.$$

Note that $\mathfrak{g}^{k+1}(\lambda)$ is codimension 1 in $\mathfrak{g}^k(\lambda)$. If $\mathfrak{g}^k(\lambda) = R(\lambda, \mathfrak{g}^k(\lambda))$, then let the sequence terminate at $\mathfrak{g}^k(\lambda)$, and set $k = d$. Thus $\mathfrak{g}^d(\lambda)$ is isotropic with respect to B_λ. If $e(\lambda) \neq \{\phi\}$, then in this way we obtain a sequence of indices $i_1, i_2, \ldots, i_d$. Note that if $R(\lambda, \mathfrak{g}) \subset \mathfrak{g}^k(\lambda)$, then $R(\lambda, \mathfrak{g}) \subset \mathfrak{g}^{k+1}(\lambda)$, $0 \leq k \leq d$; thus we have

$$R(\lambda, \mathfrak{g}) \subset R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{g}^k(\lambda), \qquad 0 \leq k \leq d.$$

Now for each $k = 1, 2, \ldots, d$, let j_k be the smallest index such that $\mathfrak{g}_{j_k} \cap \mathfrak{g}^{k-1}(\lambda) \not\subset \mathfrak{g}^k(\lambda)$.

LEMMA 3.2. *For each $k = 1, 2, 3, \ldots, d$, $i_k \in e(\lambda)$ and $j_k \in e(\lambda)$. If $k < d$, $i_k < i_{k+1}$, and for $k \leq d$, $i_k < j_k$. Moreover $\mathfrak{g}^d(\lambda) = \mathfrak{p}(\lambda)$, $i(\lambda) = \{i_k\}_{k=1}^d$, and $j(\lambda) = \{j_k\}_{k=1}^d$.*

PROOF. If $i_k \notin e(\lambda)$, there is $Y \in R(\lambda, \mathfrak{g})$ such that $\mathfrak{g}_{i_k} = \mathbf{R}Y + \mathfrak{g}_{i_k - 1}$. But since $R(\lambda, \mathfrak{g}) \subset R(\lambda, \mathfrak{g}^{k-1}(\lambda))$,

$$\mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda) = \mathbf{R}Y + (\mathfrak{g}_{i_k - 1} \cap \mathfrak{g}^{k-1}(\lambda)) \subset R(\lambda, \mathfrak{g}^{k-1}(\lambda))$$

a contradiction. If $j_k \notin e(\lambda)$, let $X \in R(\lambda, \mathfrak{g})$ such that $\mathfrak{g}_{j_k} = \mathbf{R}X + \mathfrak{g}_{j_k - 1}$. Since $R(\lambda, \mathfrak{g}) \subset \mathfrak{g}^k(\lambda)$,

$$\mathfrak{g}_{j_k} \cap \mathfrak{g}^{k-1}(\lambda) = \mathbf{R}X + (\mathfrak{g}_{j_k - 1} \cap \mathfrak{g}^{k-1}(\lambda)) \subset \mathfrak{g}^k(\lambda)$$

a contradiction. This proves the first statement of the lemma.

Now by definition of i_k,

$$\mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda) = \mathbf{R}Y + \mathfrak{g}_{i_k - 1} \cap \mathfrak{g}^{k-1}(\lambda) \subset \mathbf{R}Y + R(\lambda, \mathfrak{g}^{k-1}(\lambda)).$$

Thus

$$[\mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda), \mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda)] \subset [Y, R(\lambda, \mathfrak{g}^{k-1}(\lambda))] + [R(\lambda, \mathfrak{g}^{k-1}(\lambda)), R(\lambda, \mathfrak{g}^{k-1}(\lambda))] \subset \mathrm{Ker}(\lambda).$$

By definition of $\mathfrak{g}^k(\lambda)$ and j_k, it follows that $\mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda) \subset \mathfrak{g}^k(\lambda)$, and hence that $i_k < j_k$, and that

$$\mathfrak{g}_{i_k} \cap \mathfrak{g}^k(\lambda) = \mathfrak{g}_{i_k} \cap \mathfrak{g}^{k-1}(\lambda) \subset R(\lambda, \mathfrak{g}^k(\lambda));$$

therefore $i_k < i_{k+1}$.

Next we show that $\mathfrak{g}^d(\lambda) = \mathfrak{p}(\lambda)$. For this, let $X \in R(\lambda, \mathfrak{g}_t)$, and suppose that $X \in \mathfrak{g}^k(\lambda)$ for some $k < d$. We show that $X \in \mathfrak{g}^{k+1}(\lambda)$. Suppose $t < i_{k+1}$. By choice of i_{k+1}, $X \in \mathfrak{g}_t \cap \mathfrak{g}^k(\lambda) \subset R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{g}^{k+1}(\lambda)$. Suppose $t \geq i_{k+1}$; then $X \in R(\lambda, \mathfrak{g}_t) \cap \mathfrak{g}^k(\lambda) \subset (\mathfrak{g}_t \cap \mathfrak{g}^k(\lambda))^{\lambda, \mathfrak{g}^k(\lambda)} \subset \mathfrak{g}^{k+1}(\lambda)$. Since $X \in \mathfrak{g}^0(\lambda) = \mathfrak{g}$, it follows that $X \in \mathfrak{g}^d(\lambda)$.

Let us show now that $i(\lambda) = \{i_k\}_{k=1}^d$. Note that for each k, $R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{g}^{k+1}(\lambda)$ so that $R(\lambda, \mathfrak{g}^k(\lambda)) \subset R(\lambda, \mathfrak{g}^{k+1}(\lambda))$ and hence $R(\lambda, \mathfrak{g}^k(\lambda)) \subset \mathfrak{p}(\lambda)$. Thus $\mathfrak{g}_{i_k} \subset \mathfrak{g}_{i_k - 1} + \mathfrak{p}(\lambda)$ and $i_k \notin j(\lambda)$. It follows that $i(\lambda) = \{i_k\}_{k=1}^d$.

Finally, to see that $\{j_k\}_{k=1}^d = j(\lambda)$, note that $j_k \in j(\lambda)$, since if not, then $\mathfrak{g}_{j_k} \subset \mathfrak{g}_{j_k - 1} + \mathfrak{g}^k(\lambda)$, hence $\mathfrak{g}_{j_k} \cap \mathfrak{g}^{k-1}(\lambda) \subset (\mathfrak{g}_{j_k - 1} + \mathfrak{g}^k(\lambda)) \cap \mathfrak{g}^{k-1}(\lambda) = \mathfrak{g}_{j_k - 1} \cap \mathfrak{g}^{k-1}(\lambda) + \mathfrak{g}^k(\lambda) = \mathfrak{g}^k(\lambda)$ a contradiction.

Let $j \in j(\lambda)$, and let k_0 be the smallest k, $1 \leq k \leq d$, such that $\mathfrak{g}_j \subset \mathfrak{g}_{j-1} + \mathfrak{g}^k(\lambda)$. We claim that $j = j_{k_0}$. Now $\mathfrak{g}_j \subset \mathfrak{g}_{j-1} + \mathfrak{g}^{k_0 - 1}(\lambda)$, so we may write $\mathfrak{g}_j = \mathbf{R}X + \mathfrak{g}_{j-1}$ where $X \in \mathfrak{g}^{k_0 - 1}(\lambda)$, and by choice of k_0, $X \in \mathfrak{g}^{k_0}(\lambda)$. Hence $\mathfrak{g}_j \cap \mathfrak{g}^{k_0 - 1}(\lambda) \subset \mathfrak{g}^{k_0}(\lambda)$ and $j \geq j_{j_{k_0}}$, by choice of $j \geq j_{j_{k_0}}$. If $j > j_{k_0}$, then choose $\tilde{X} \in \mathfrak{g}_{j_{k_0}}$ such that $\mathfrak{g}^{k_0 - 1}(\lambda) = \mathbf{R}\tilde{X} + \mathfrak{g}^{k_0}(\lambda)$. Since $\dim(\mathfrak{g}^{k_0 - 1}(\lambda)/\mathfrak{g}^{k_0}(\lambda)) = 1$, there are elements $a \neq 0$, $b \neq 0$ in such that $W = a\tilde{X} + bX \in \mathfrak{g}^{k_0}(\lambda)$. But $\mathfrak{g}_j = \mathbf{R}W + \mathfrak{g}_{j-1} \subset \mathfrak{g}_{j-1} + \mathfrak{g}^{k_0}(\lambda)$, contradicting our choice of k_0; therefore $j = j_{k_0}$, and the proof of the lemma is finished. □

Now let E denote the set of pairs

$$E = \{(e(\lambda), j(\lambda)) | \lambda \in \mathfrak{g}^*\}$$

and for d a positive integer, let

$$E_d = \{(e, j) \in E | \operatorname{card}(j) = d\}.$$

Let us regard elements of E_d as ordered $3d$-tuples of integers

$$(e, j) = (e_1, e_2, \ldots, e_{2d}, j_1, j_2, \ldots, j_d)$$

where $e_1 < e_2 < \cdots$, and $\{j_1, \ldots, j_d\} = j$ is indexed by the inductive process above. We define a total order on E in the following way. Let (ϕ, ϕ) be the maximal element of E, and regarding E_d as above, let E_d have the natural lexographic ordering. If $d > d'$, let us say that for any $\alpha \in E_d$, $\alpha' \in E_{d'}$, $\alpha < \alpha'$.

Now for each $\alpha \in E$, set $\Omega_\alpha = \{\lambda \in \mathfrak{g}^* | (e(\lambda), j(\lambda)) = \alpha\}$ and for each $e_0 = e(\lambda_0)$, $\Omega_{e_0} = \{\lambda \in \mathfrak{g}^* | e(\lambda) = e_0\}$ so that $\Omega_{e_0} = \bigcup\{\Omega_{(e,j)} | e = e_0\}$. If $s \in G$, and $\lambda \in \mathfrak{g}^*$, then $\mathfrak{g}^k(\mathrm{Ad}^*(s)\lambda) = \mathrm{Ad}(s)(\mathfrak{g}^k(\lambda))$ and it follows that each $\alpha \in E$, Ω_α is *G-invariant.* The sets Ω_{e_0} were first considered by Pukanszky in [**10**], and the sets Ω_α are considered by N. V. Pedersen in a paper to appear.

Let $\{Z_1, Z_2, \ldots, Z_p\}$ be a basis compatible with the Jordan-Hölder sequence chosen at the beginning of this section. Let $e = e(\lambda_0)$ for some λ_0, let $P_e^{ij}(\lambda) = \lambda([Z_{e_i}, Z_{e_j}])$, $e_i, e_j \in e$, and set $P_e(\lambda) = \det((P_e^{ij}(\lambda)))$. Letting the set $\{e(\lambda) | \lambda \in \mathfrak{g}^*\}$ have the total ordering inherited from E, it is shown in [9] that

$$\Omega_e = \{\lambda \in \mathfrak{g}^* | P_{e'}(\lambda) = 0,\ e' < e \text{ and } P_e(\lambda) \neq 0\}.$$

Now for each $e = e(\lambda)$, let $J_e = \{j | (e,j) \in E\}$, and let J_e have the total ordering inherited from E.

PROPOSITION 3.3. *There are polynomials $P_{(e,j)}$, $j \in J_e$, such that for each $j \in J_e$,*

$$\Omega_{(e,j)} = \{\lambda \in \Omega_e | P_{(e,j')}(\lambda) = 0, j' < j \text{ and } P_{(e,j')}(\lambda) \neq 0\}.$$

PROOF. Let $j \in J_e$, and write $j = \{j_1, \ldots, j_d\}$ and $i = e - j = \{i_1, \ldots, i_d\}$ as in the inductive process above. Let $\lambda \in \Omega_e$ and for each $k = 1, 2, \ldots, d$ set $e^{(k)} = e - \{i_1, \ldots, i_k, j_1, \ldots, j_k\}$ and define elements $Z_t^k(\lambda) \in g$, $t \in e^{(k)}$ as follows. Note that $\{Z_t\}_{t\in e}$ is a basis for $\mathfrak{g}$ modulo $R(\lambda, \mathfrak{g})$. Let $Z_t^1(\lambda) = Z_t$ if $t \in e^{(1)}$, $t < j_1$ and for $t > j_1$, set

$$Z_t^1(\lambda) = B_\lambda(Z_{j_1}, Z_{i_1})Z_t - B_\lambda(Z_t, Z_{i_1})Z_{j_1}.$$

Suppose that $\lambda \in \Omega_{(e,j')}$ with $j' \geq j$ and write $j' = \{j_1', \ldots, j_d'\}$, $i = e - j' = \{i_1', \ldots, i_d'\}$. Since $e_1 = i_1 = i_1'$, clearly $j_1' = j_1$ if and only if $B_\lambda(Z_{j_1}, Z_{i_1}) \neq 0$, and in this case, $\{Z_t^1(\lambda)\}_{t\in e^{(1)}}$ is a basis of $\mathfrak{g}^1(\lambda)$ modulo $R(\lambda, \mathfrak{g}^1(\lambda))$. Therefore, by definition of j_1, j_2, we have $j_1' = j_1$ and $j_2' = j_2$ if and only if, $B_\lambda(Z_{j_1}, Z_{i_1}) \neq 0$ and $B_\lambda(Z_{j_2}^1(\lambda), Z_{i_2}^1(\lambda)) \neq 0$. Now define $Z_t^2(\lambda)$, $t \in e^{(2)}$ in the same way as $\{Z_t^1(\lambda)\}_{t\in e^{(1)}}$, so that if $j_1' = j_1$, $j_2' = j_2$, then $\{Z_t^2(\lambda)\}_{t\in e^{(2)}}$ is a basis of $\mathfrak{g}^2(\lambda)$ modulo $R(\lambda, \mathfrak{g}^2(\lambda))$. Continuing in this way, set

$$P_{e,j}(\lambda) = B_\lambda(Z_{j_1}, Z_{i_1}) B_\lambda(Z_{j_2}^1(\lambda), Z_{i_2}^1(\lambda)) \cdots B_\lambda(Z_{j_d}^{d-1}(\lambda), Z_{i_d}^{d-1}(\lambda))$$

and the proposition follows. Q.E.D.

COROLLARY 3.4. *Let α_0 be the minimal element of E. Then Ω_{α_0} is Zariski open in $\mathfrak{g}^*$. Moreover, for each $\alpha \in E$, Ω_α is open in $\bigcup_{\beta\geq\alpha} \Omega_\beta$.*

PROOF. That Ω_{α_0} is Zariski open is clear. As for the second statement, let $\alpha \in E$, $\alpha = (e,j)$. Ω_α is Zariski open in $\bigcup_{j'\geq j} \Omega_{(e,j')}$ by Proposition 3.3. But Ω_e is open in $\bigcup_{e'\geq e} \Omega_{e'}$, and hence $\bigcup_{j'\geq j} \Omega_{(e,j')} = \Omega_e \cap \bigcup_{\beta\geq\alpha} \Omega_\beta$ is open in $\bigcup_{\beta\geq\alpha} \Omega_\beta$, and it follows that Ω_α is open in $\bigcup_{\beta\geq\alpha} \Omega_\beta$. Q.E.D.

Now for each $\lambda \in \mathfrak{g}^*$ such that $\{\phi\} \neq e(\lambda) = i(\lambda) \cup j(\lambda)$, $i(\lambda) = \{i_1, i_2, \ldots, i_d\}$, $j(\lambda) = \{j_1, j_2, \ldots, j_d\}$, define for each $0 \leq k < d$, $\mathfrak{m}_{k1}(\lambda) = \mathfrak{g}^k(\lambda) \cap \mathfrak{g}_{i_{k+1}}$ and $\mathfrak{m}_{k0}(\lambda) = \mathfrak{g}^k(\lambda) \cap \mathfrak{g}_{i_{k+1}-1}$. Then $(\mathfrak{m}_{k1}(\lambda), \mathfrak{m}_{k0}(\lambda))$ is a Kirillov pair in $\mathfrak{g}^k(\lambda)$ at λ, and $\mathfrak{m}_{k1}^\lambda \mathfrak{g}^{k(\lambda)} = \mathfrak{g}^{k+1}(\lambda)$, $0 \leq k < d$. Thus the sequence $\mathfrak{g}^0(\lambda) \supset \mathfrak{g}^1(\lambda) \supset \cdots \supset \mathfrak{g}^d(\lambda)$ is a Kirillov sequence for λ in $\mathfrak{g}$.

THEOREM 3.5. *Let d be a positive integer and let $\alpha \in E_d$. Let $\{\lambda_n\}_{n=0}^\infty$ be a sequence in Ω_α which converges to λ_0. Then for each $n \geq 0$, there is an irreducible representation π_n corresponding to λ_n such that if*

$$D = P(t_1, t_2, \ldots, t_d, \partial/\partial t_1, \partial/\partial t_2, \ldots, \partial/\partial t_d)$$

is any polynomial differential operator, then there is an integer $m > 0$ *and a sequence* $\{W_n\}_{n=0}^{\infty}$ *in* $U(\mathfrak{g}_c)^{(m)}$ *which converges to* W_0 *and such that* $\pi_n(W_n) = D$, $n = 0, 1, 2, \ldots$.

PROOF. Clearly we may assume that for some $1 \le k \le d$, either $D = t_k$ or $D = \partial/\partial t_k$. For each $n \ge 0$, we have the data $\{\mathfrak{g}^k(\lambda_n)\}_{k=0}^d$, $\{(\mathfrak{m}_{k1}(\lambda_n), \mathfrak{m}_{k0}(\lambda_n))\}_{k=0}^{d-1}$ as in the remarks preceding the theorem. Note that for each $n \ge 0$, $1 \le t \le p$, and $1 \le k \le d$, $\dim(\mathfrak{g}_t \cap \mathfrak{g}^k(\lambda_n)) = \mathrm{card}(\{j_s | s \le k, j_s < t\})$. Now by Lemma 1.1, we have that $\mathfrak{g}^1(\lambda_n) \to \mathfrak{g}^1(\lambda_0)$. Since $\dim_{\mathbf{R}}(\mathfrak{g}_t \cap \mathfrak{g}^1(\lambda_n)) = \dim_{\mathbf{R}}(\mathfrak{g}_t \cap \mathfrak{g}^1(\lambda_0))$, $n = 1, 2, 3, \ldots$, it follows that $\mathfrak{g}_t \cap \mathfrak{g}^1(\lambda_n) \to \mathfrak{g}_t \cap \mathfrak{g}^1(\lambda_0)$, and in particular, $\mathfrak{m}_{11}(\lambda_n) \to \mathfrak{m}_{11}(\lambda_0)$ and $\mathfrak{m}_{10}(\lambda_n) \to \mathfrak{m}_{10}(\lambda_0)$. But then Lemma 1.1 implies that $\mathfrak{g}^2(\lambda_n) \to \mathfrak{g}^2(\lambda_0)$. Continuing in this way, we obtain for each $k = 0, 1, \ldots, d-1$, $\mathfrak{g}^{k+1}(\lambda_n) \to \mathfrak{g}^{k+1}(\lambda_0)$, $\mathfrak{m}_{k1}(\lambda_n) \to \mathfrak{m}_{k1}(\lambda_0)$ and $\mathfrak{m}_{k0}(\lambda_n) \to \mathfrak{m}_{k0}(\lambda_0)$. Now, for each $0 \le k \le d$, $n \ge 0$, we shall define an irreducible representation $\pi_k^{(n)}$ of $G^k(\lambda_n) = \exp_G(\mathfrak{g}^k(\lambda_n))$. Let $\pi_d^{(n)}$ be the character of $G^d(\lambda_n)$ with differential $i(\lambda_n | \mathfrak{g}_d(\lambda_n))$. Choose $X_d^{(0)} \in \mathfrak{g}^{d-1}(\lambda_0) \sim \mathfrak{g}^d(\lambda_0)$, and since $\mathfrak{g}^{d-1}(\lambda_n) \to \mathfrak{g}^{d-1}(\lambda_0)$ and $\mathfrak{g}^d(\lambda_n) \to \mathfrak{g}^d(\lambda)$, we can choose $X_d^{(n)} \in \mathfrak{g}^{d-1}(\lambda_n) \sim \mathfrak{g}^d(\lambda_n)$, $n = 1, 2, 3, \ldots$, such that $X_d^{(n)} \to X_d^{(0)}$. Now for each n, define $\pi_{d-1}^{(n)} = \pi(\pi_d^{(n)}, X_d^{(n)})$ as in formula (1) above, that is, for each $f \in L^2(\mathbf{R}, H(\pi_d^{(n)})) = L^2(\mathbf{R})$, $y \in G^d(\lambda_n)$, and $s, t \in \mathbf{R}$,

$$(\pi_{d-1}^{(n)}(y \cdot \exp_G(sX_d^{(n)}))f)(t) = \pi_d^{(n)}(\exp_G(tX_d^{(n)}) \cdot y \cdot \exp(-tX_d^{(n)}))f(t+s).$$

We continue in this way, choosing $X_k^{(0)} \in \mathfrak{g}^{k-1}(\lambda_0) \sim \mathfrak{g}^k(\lambda_0)$ and $X_k^{(n)} \in \mathfrak{g}^{k-1}(\lambda_n) \sim \mathfrak{g}^k(\lambda_n)$, $n = 1, 2, 3, \ldots$, such that $X_k^n \to X_k^{(0)}$, for each k, so that

$$H(\pi_k^{(n)}) = L^2(\mathbf{R}, H(\pi_{k+1}^{(n)})), \qquad n \ge 0.$$

For each $k < d$, denote elements of $\mathbf{R}^{d-k}$ by $(t_{k+1}, t_{k+2}, \ldots, t_d)$, set $U_{d-1} =$ identity mapping on $L^2(\mathbf{R})$, and define for $k < d-1$, $U_k \colon H(\pi_k) \to L^2(\mathbf{R}^{d-k})$ by $U_k f(t_{k+1}, t_{k+2}, \ldots, t_d) = U_{k+1}(f(t_{k+1}))(t_{k+2}, \ldots, t_d)$. Set for each $n \ge 0$, $\pi_n = U_0 \pi_0^{(n)} U_0^{-1}$. Now suppose that $D = t_k$, and set $j = k-1$. For each $n \ge 0$, let $Y_k^{(n)} \in \mathfrak{m}_{j1}(\lambda_n) \sim \mathfrak{m}_{j0}(\lambda_n)$ such that $\lambda(Y_k^{(n)}) = 0$ and $B_{\lambda_n}(X_k^{(n)}, Y_k^{(n)}) = 1$. It is easily seen that $Y_k^{(n)} \to Y_k^{(0)}$, and that for each n, $U_j \pi_j^{(n)} U_j^{-1}(-iY_k^{(n)}) = t_k$. If $j = 0$, we are done. Otherwise, we apply Lemma 2.3 to obtain, for each n, $W^{(n)} \in U(g^k(\lambda_n)_c)$ such that $\pi_{j-1}^{(n)}(W^{(n)}) = \pi_j^{(n)}(-iY_k^{(n)})$. The construction whereby $W^{(n)}$ is obtained involves only $\mathrm{ad}\, X_j^{(n)}$, $Y_j^{(n)} = Y_{k-1}^{(n)} \in \mathfrak{m}_{k1}(\lambda_n) \cap \ker(\lambda_n) \sim \mathfrak{m}_{k0}(\lambda_n)$ where $B_{\lambda_n}(X_j^{(n)}, Y_j^{(n)}) = 1$, $n = 0, 1, 2, 3, \ldots$, and we have $Y_j^{(n)} \to Y_j^{(0)}$ as well as $\mathrm{ad}\, X_j^{(n)} \to \mathrm{ad}\, X_j^{(0)}$. Hence it is clear that for some m, $W^{(n)} \in U(\mathfrak{g}_c)^{(m)}$, $n \ge 0$, and $W^{(n)} \to W^{(0)}$, and from the definition of U_{j-1} it is clear that for each n, $U_{j-1} \pi_{j-1}^{(n)} U_{j-1}^{-1}(W^{(n)}) = t_k$. If $j = 1$, then we are done. If $j > 1$, then we continue this process applying Lemma 2.3 at each step. This finishes the case $D = t_k$. If $D = \partial/\partial t_k$, the proof is similar. Q.E.D.

4. A theorem. We now drop the assumption that $\mathfrak{g}$ is nilpotent, that is, let $\mathfrak{g}$ be a real solvable Lie algebra of exponential type, and G a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, and for the remainder of this paper, let $\mathfrak{n} = \mathfrak{n}_p \supset \mathfrak{n}_{p-1} \supset \cdots \supset \mathfrak{n}_0 = (0)$ be a Jordan-Hölder sequence for $\mathfrak{n}$ having the property that for each $t = 1, 2, 3, \ldots, p-1$, if $[\mathfrak{g}, \mathfrak{n}_t] \not\subset \mathfrak{n}_t$, then $[\mathfrak{g}, \mathfrak{n}_{t+1}] \subset \mathfrak{n}_{t+1}$. Let E be the index set and $\{\Omega_\alpha\}_{\alpha \in E}$ the $\mathrm{Ad}^*(N)$-invariant partition of $\mathfrak{n}^*$ corresponding to this Jordan-Hölder sequence as constructed in the previous section.

Let $\lambda \in \mathfrak{n}^*$, $\mathfrak{p}(\lambda) = \sum_t R(\lambda, \mathfrak{n}_t)$. Suppose that there is $A \in R(\lambda, \mathfrak{g}) \sim \mathfrak{n}$, and set $\mathfrak{h} = \mathbf{R}A + \mathfrak{n}$, $H = \exp(\mathfrak{h})$. It is shown in [**1**] that $\mathfrak{p}(\lambda)$ is invariant under $\mathrm{ad}\, A$, and we may extend the equivalence class of $\sigma = \mathrm{ind}(\lambda, \mathfrak{p}(\lambda))$ in $\hat{N}$ to H by setting

$$(2)\quad (\sigma(\exp sA)f)(y) = f(\exp(-sA) \cdot y \cdot \exp sA) \exp^{\{\frac{1}{2} \mathrm{tr}(\mathrm{ad}_{\mathfrak{n}/\mathfrak{p}(\lambda_n)} A_n)\}}, \qquad y \in N.$$

The corresponding extension of λ to $\mathfrak{h}$ is obtained by setting $\lambda(A) = 0$. Now let $\mathfrak{n} = \mathfrak{n}^0(\lambda) \supset \mathfrak{n}^1(\lambda) \supset \cdots \supset \mathfrak{n}^d(\lambda) = \mathfrak{p}(\lambda)$ be the Kirillov sequence in λ as constructed in the previous section, let $X_k \in \mathfrak{n}^{k-1}(\lambda) \sim \mathfrak{n}^k(\lambda)$, $k = 1, 2, \ldots, d$, and let $\pi = \pi_0$ be the irreducible representation of N corresponding to λ as constructed in Theorem 3.5. Then $\pi_0 = U\sigma U^{-1}$ where $U\colon H(\pi_0) \to L^2(\mathbf{R}^d)$ is defined by

$$Uf(t_1, t_2, \ldots, t_d) = f(\exp t_1 X_1 \cdot \exp t_2 X_2 \cdots \exp t_d X_d).$$

We extend π_0 as indicated above (that is, so as to be isomorphic with the above extension of σ).

Now let $\alpha \in E$ such that $\lambda \in \Omega_\alpha$, and suppose that $\{\lambda_n\}_{n=1}^\infty$ is a sequence in Ω_α such that $\lambda_n \to \lambda$. By Theorem 3.5, we have a corresponding sequence $\{\pi_n\}_{n=1}^\infty$ of irreducible representations of N such that if D is a polynomial differential operator on $\mathbf{R}^d$, then there is $W_n \in U(\mathfrak{n}_c)^m$, $n = 0, 1, 2, \ldots$, for some m, such that $W_n \to W_0$ and $\pi_n(W_n) = D$ for each n. Recall that for each n, we have $X_k^{(n)} \in \mathfrak{n}^{k-1}(\lambda_n) \sim \mathfrak{n}^k(\lambda_n)$, $k = 1, 2, \ldots, d$ such that $\mathrm{ind}(\lambda, \mathfrak{p}(\lambda_n))$ is equivalent to π_n via the isomorphism

$$Uf(t_1, t_2, \ldots, t_d) = f(\exp t_1 X_1^{(n)} \cdot \exp t_2 X_2^{(n)} \cdots \exp t_d X_d^{(n)})$$

and for each k, $X_k^{(n)} \to X_k$. Suppose that we have $A_n \in R(\lambda_n, \mathfrak{g}) \sim \mathfrak{n}$, $n = 1, 2, 3, \ldots$, such that $A_n \to A$, and set $h_n = \mathbf{R}A_n + \mathfrak{n}$, $H_n = \exp h_n$ for each n. Extend π_n to H_n, as above, so that the corresponding extension of λ_n is obtained by setting $\lambda_n(A_n) = 0$. Let the algebra $\mathfrak{D}$ of polynomial differential operators on $\mathbf{R}^d$ have the (obvious) filtration $\mathfrak{D}^{(0)} \subset \mathfrak{D}^{(1)} \subset \mathfrak{D}^{(2)} \subset \cdots$ so that $D \in \mathfrak{D}^{(m)}$ if and only if there is a polynomial P of degree $\leq m$ such that $D = P(t_1, t_2, \ldots, t_d, \partial/\partial t_1, \partial/\partial t_2, \ldots, \partial/\partial t_d)$, and for each m, let $\mathfrak{D}^{(m)}$ have the usual topology as a finite dimensional vector space over $\mathbf{C}$.

LEMMA 4.1. *There is an integer $m > 0$ such that $\pi_0(A) \in \mathfrak{D}^{(m)}$, $\{\pi_n(A_n)\}_{n=0}^\infty \subset \mathfrak{D}^{(m)}$, and $\pi_n(A_n) \to \pi_0(A)$ in $\mathfrak{D}^{(m)}$.*

PROOF. Clearly we may assume that $d > 0$. Let us use the notation $T = (t_1, t_2, \ldots, t_d)$, $U = (u_1, u_2, \ldots, u_p)$ and $Z = (z_{11}, z_{12}, \ldots, z_{ij}, \ldots, z_{pp})$ for elements of $\mathbf{R}^d$, $\mathbf{R}^p$, and $\mathbf{R}^{p^2}$, respectively, and denote the objects associated with π_0 by λ_0, A_0, etc. For each $n \geq 0$, let $\{X_k^{(n)}\}_{k=d+1}^p$ be a basis of $\mathfrak{p}(\lambda_n)$ such that $X_k^{(n)} \to X_k^{(0)}$, $d < k \leq p$, and for each $i, j = 1, 2, \ldots, p$ and $s \in \mathbf{R}$, let $a_{ij}^{(n)}(s)$ denote the coefficient of $X_j^{(n)}$ in the expansion of $e^{s\,\mathrm{ad}A_n}(X_i^{(n)})$ in terms of the ordered basis $X_1^{(n)}, X_2^{(n)}, \ldots, X_p^{(n)}$ of $\mathfrak{g}$. Denote the element $(a_{11}^{(n)}(s), a_{12}^{(n)}(s), \ldots, a_{ij}^{(n)}(s), \ldots, a_{pp}^{(n)}(s))$ of $\mathbf{R}^{p^2}$ by $a^{(n)}(s)$.

By the Campbell-Hausdorff formula, we have for each n, polynomials $P_1^{(n)}, P_2^{(n)}, \ldots, P_p^{(n)}$ in T such that

$$\prod_{j=1}^{p} \exp t_j X_j^{(n)} = \exp\left(\sum_{j=1}^{p} P_j^{(n)}(T) X_j^{(n)}\right).$$

Let $q > 0$ and such that N is step q. Then for each n, j, $\deg(P_j^{(n)}) < q$, and the coefficients of $P_j^{(n)}$ depend only on the structure constants $(b_k^{ij})^{(n)}$, $[X_i^{(n)}, X_j^{(n)}] = \sum (b_k^{ij})^{(n)} X_k^{(n)}$. Clearly for each i, j, k, $(b_k^{ij})^{(n)} \to (b_k^{ij})^{(0)}$ and hence $P_j^{(n)} \to P_j^{(0)}$ in the vector space $\mathbf{C}[T]^{(q)}$, $1 \le j \le p$. Now let the polynomials $\tilde{P}_j^{(n)}$ in T and Z be defined by $\tilde{P}_j^{(n)}(T, Z) = \sum_i P_i^{(n)}(T) z_{ij}$; then we have

$$\exp -sA_n \left[\prod_{j=1}^{d} \exp t_j X_j^{(n)}\right] \exp sA_n = \exp \sum_{j=1}^{p} \tilde{P}_j^{(n)}(T, a^{(n)}(s)) X_j^{(n)},$$
$$s \in \mathbf{R},\ n = 0, 1, 2, 3, \ldots.$$

On the other hand, there are polynomials $R_j^{(n)}$, $1 \le j \le p$, in U such that

$$\exp \sum_{j=1}^{p} u_j X_j^{(n)} = \exp \sum_{j>d} R_j^{(n)}(U) X_j^{(n)} \cdot \prod_{j=1}^{d} \exp R_j^{(n)}(U) X_j^{(n)}.$$

As with $P_j^{(n)}$, we see that for each n, j, $\deg(R_j^n) \le q$, and for each j, $R_j^{(n)} \to R_j^{(0)}$ in $\mathbf{C}[U]^{(q)}$. Now let $Q_j^{(n)} = R_j^{(n)}(\tilde{P}_1^{(n)}, \ldots, \tilde{P}_p^{(n)})$, $1 \le j \le p$. Then $Q_j^{(n)} \to Q_j^{(0)}$ in $\mathbf{C}[T, Z]^{(q^2)}$ and from the definition of π_n we have, for each $\phi \in C^\infty(\pi_n)$,

$$\begin{aligned}(\pi_n(A_n)\phi)(T) = \frac{d}{ds}\bigg|_{s=0} & \exp^{i\sum_{i>d} Q_j^{(n)}(T, a^{(n)}(s))\lambda_n(X_j^{(n)})} \\ & \cdot \phi\left(Q_1^{(n)}(T, a^{(n)}(s)), Q_2^{(n)}(T, a^{(n)}(s)), \ldots, Q_d^{(n)}(T, a^{(n)}(s))\right) \\ & \cdot \exp^{\{-\frac{1}{2}\operatorname{tr}(\mathrm{ad}_{n/p(\lambda_n)} A_n)\}}\end{aligned}$$

for each n. Let $\tilde{Q}_j^{(n)}$ be the polynomial in $\mathbf{C}[T]^{(q)}$ such that

$$\frac{d}{ds}\bigg|_{s=0} Q_j^{(n)}(T, a^{(n)}(s)) = \tilde{Q}_j^{(n)}(T), \qquad 1 \le j \le p,\ n \ge 0.$$

Note that $a_{ij}^{(n)}(0) = \delta_{ij}$ for each i, j and n, and for each i, j, $d/ds|_{s=0}\, a_{ij}^{(n)}(s) \to d/ds|_{s=0}\, a_{ij}^{(0)}(s)$, whence $\tilde{Q}_j^{(n)} \to \tilde{Q}_j^{(0)}$. Since for each n,

$$\pi_n(A_n) = i\sum_{j>d} \tilde{Q}_j^{(n)}(T)\lambda_n(X_j^{(n)}) + \sum_{j=1}^{d} \tilde{Q}_j^{(n)}(T)\frac{\partial}{\partial t_j} - \frac{1}{2}\operatorname{tr}(\mathrm{ad}\mathfrak{n}/P(\lambda_n)A_n)$$

the result follows. Q.E.D.

For each $n > 0$, define A_n^* in $\mathfrak{h}_n^*$ by setting $A_n^*(A_n) = 1$, $A_n^*|_{\mathfrak{n}} \equiv 0$.

COROLLARY 4.2. *There is an integer $m > 0$ and for each $n = 0, 1, 2, \ldots$, there is $W_n \in U((\mathfrak{h}_n)_c)$ such that $\{W_n\}_{n=0}^{\infty} \subset U(\mathfrak{g}_c)^{(m)}$, $W_n \to W_0$ in $U(\mathfrak{g}_c)^{(m)}$, and such that for any real sequence $\{c_n\}_{n=0}^{\infty}$, $(\chi_{c_n} \otimes \pi_n)(W_n) = c_n$ where χ_{c_n} is the character of H_n with differential $ic_n A_n^*$.*

PROOF. By Lemma 4.1, for each n we may write $\pi_n(A_n) = \sum_\mu a_\mu^{(n)} D_\mu$ where $\{D_\mu\}$ is a finite collection of polynomial differential operators and for each μ, $\{a_\mu^{(n)}\}_{n=0}^{\infty}$ is a sequence of complex numbers such that $a_\mu^{(0)} = \lim_n a_\mu^{(n)}$. By Theorem 3.5, for each μ, there is a sequence $\{V_\mu^{(n)}\}_{n=0}^{\infty} \subset U(\mathfrak{n}_c)^{(m_\mu)}$ such that $V_\mu^{(n)} \to V_\mu^{(0)}$ in $U(\mathfrak{n}_c)^{(m_\mu)}$ and such that for each n, $\pi_n(V_\mu^{(n)}) = D_\mu$. Thus $\pi_n(A_n - \sum_\mu a_\mu^{(n)} V_\mu^{(n)}) = 0$, $n = 0, 1, 2, \ldots$, and we may take $m = \max_\mu\{m_\mu\}$ and

$$W_n = -i\left(A_n - \sum_\mu a_\mu^{(n)} V_\mu^{(n)}\right), \qquad n = 0, 1, 2, \ldots. \quad \text{Q.E.D.}$$

Let $K(G)$ be the space of all closed, connected subgroups of G (with the compact-open topology), and let $S(G)$ be the space of all pairs (ρ, H) where $H \in K(G)$ and ρ is an unitary equivalence class of representations of H with the topology of Fell (cf. [**5**]). Let $K_N(G)$ be the set of all $H \in K(G)$ such that $N \subset H$, and $S_N(G)$ the set of all $(\rho, H) \in S(G)$ such that $H \in K_N(G)$ and $\rho \in \hat{H}$. For each $H \in K_N(G)$ we have a topological embedding of $\hat{H}$ in $S_N(G)$. The proof that η_G is continuous (cf. [**10**, Proposition 2]) is easily generalized to show that the mapping $\Theta\colon \mathfrak{g}^* \times K_N(G) \to S_N(G)$ given by

$$\Theta((l, H)) = (\eta_H(\mathrm{Ad}^*(H))(l|_{\mathfrak{h}}), H)$$

is continuous, where $\mathfrak{g}^* \times K_N(G)$ has the product topology. If $(\rho, H) \in S_N(G)$ we denote the $\mathrm{Ad}^*(H)$-orbit $\eta_H^{-1}(\rho)$ by O_ρ, and if $J \subset H$, $\mathfrak{j} = \log(J)$, let $O_\rho|_{\mathfrak{j}} = \{l|_{\mathfrak{j}} \mid l \in O_\rho\}$.

The following two facts are well-known consequences of the general theory (cf. [**4** and **5**]).

LEMMA 4.3. *Let $(\rho, H) \in S_N(G)$ and let $J \in K_N(G)$ be a subgroup of H. Then the set of all $\sigma \in \hat{J}$ such that $O_\sigma \subset O_\rho|_{\log(J)}$ is a dense subset of* $\mathrm{Sp}(\rho|_J)$.

LEMMA 4.4. *Let $(\rho, H) \in S_N(G)$, and let $\{(\rho_n, H_n)\}_{n=1}^{\infty}$ be a sequence in $S_N(G)$ such that $(\rho_n, H_n) \to (\rho, H)$. Let $J \in K_N(G)$ and for each n, $J_n \in K_N(G)$ such that $J \subset H$, $J_n \subset H_n$, and $J_n \to J$. Let $(\sigma, J) \in S_N(G)$ such that $O_\sigma \subset O_\rho|_{\log(J)}$. Then for each n, there is $\sigma_n \in \hat{J}_n$ such that $O_{\sigma_n} \subset O_{\rho_n}|_{\log(J_n)}$ and such that the sequence $\{(\sigma_n, J_n)\}_{n=1}^{\infty}$ converges to (σ, J).*

We define a partition of $S_N(G)$ as follows. For each $(\rho, H) \in S_N(G)$, let $\alpha(\rho)$ be the smallest index in E such that $O_\rho|_{\mathfrak{n}} \cap \Omega_\alpha \neq \{\phi\}$. For each $\alpha \in E$, let

$$\tilde{V}_\alpha = \{(\rho, H) \in S_N(G) | \alpha(\rho) = \alpha\}.$$

From Brown's Theorem [**3**] and Lemma 4.3 above it follows that $(\rho, H) \in \tilde{V}_\alpha$ if and only if $\mathrm{Sp}(\rho|_N) \cap \eta_N(\Omega_\alpha) \neq \{\phi\}$ and $\mathrm{Sp}(\rho|_N) \cap \eta_N(\Omega_\beta) = \{\phi\}$ for all $\beta < \alpha$. For each α, set $V_\alpha = \tilde{V}_\alpha \cap \hat{G}$, and $U_\alpha = \eta_G^{-1}(V_\alpha)$. Then $U_\alpha = \{O \in \mathfrak{g}^*/\mathrm{Ad}^*(G) | O|_{\mathfrak{n}} \cap \Omega_\alpha \neq \{\phi\}$ and $O|_{\mathfrak{n}} \cap \Omega_\beta = \{\phi\}$, for all $\beta < \alpha\}$.

LEMMA 4.5. *For each* $\alpha, \tilde{V}_\alpha$ *is open in* $\bigcup_{\beta\geq\alpha}\tilde{V}_\beta$. *If* α_0 *is the smallest element of* E, *then* $\tilde{V}_{\alpha_0}$ *is dense in* $S_N(G)$.

PROOF. Let $(\rho, H) \in \tilde{V}_\alpha$ and suppose that $\{(\rho_n, H_n)\}_{n=1}^\infty$ is a sequence in $\bigcap_{\beta\geq\alpha}\tilde{V}_\rho$ such that $(\rho_n, H_n) \to (\rho, H)$. Let $\sigma_0 \in \mathrm{Sp}(\rho|_N)$ such that $O_{\sigma_n} \subset O_\rho|_{\mathfrak{n}} \cap \Omega_\alpha$. By Lemma 4.4, there is $\{\sigma_n\} \subset \hat{N}$ such that $\sigma_n \to \sigma$ and for each n, $O_{\sigma_n} \subset O_{\rho_n}|_N$. By Brown's Theorem, $O_{\sigma_n} \to O_\sigma$. Since $\{O_{\sigma_n}\} \subset \bigcup_{\beta\geq\alpha}\Omega_\alpha$ and Ω_α is open in $\bigcup_{\beta\geq\alpha}\Omega_\beta$, $\{O_{\sigma_n}\}$ is eventually in Ω_α, thus $\{(\rho_n, H_n)\}_{n=1}^\infty$ is eventually in $\tilde{V}_\alpha$.

Let α_0 be the minimal element of E and let $(\rho, H) \in S_N(G)$, $\mathfrak{h} = \log(H)$. The set $\{O \in \mathfrak{h}^*/\mathrm{Ad}^*(H) |\ O|_{\mathfrak{n}} \cap \Omega_{\alpha_0} \neq \{\phi\}\}$ is dense in $\mathfrak{h}^*/\mathrm{Ad}^*(H)$, hence the set $\{\rho \in \hat{H} |\ O_\rho|_{\mathfrak{n}} \cap \Omega_\alpha \neq \{\phi\}\}$ is dense in $\hat{H}$ (by continuity of η_H). The embedding of this set in $S_N(G)$ is contained in $\tilde{V}_{\alpha_0}$ and (ρ, H) is contained in its closure. □

Let $(\rho, H) \in S_N(G)$, $\mathfrak{h} = \log(H)$, and let $\lambda \in O_\rho|_{\mathfrak{n}}$. Let $\mathfrak{j} = R(\lambda, h) + \mathfrak{n}$, $J = \exp(\mathfrak{j})$, and let $\sigma \in \hat{J}$ such that $O_\sigma \subset O_\rho|_{\mathfrak{j}}$. Suppose that $\mathfrak{j} \neq \mathfrak{n}$, and let $\{A_1, A_2, \dots, A_r\} \subset R(\lambda, h)$ be a basis for $\mathfrak{j}$ mod $\mathfrak{n}$. Define $A_j^* \in \mathfrak{h}^*$ by $A_j^*(A_i) = \delta_{ij}$ and $A_j^*|_{\mathfrak{n}} \equiv 0$. Recall then that $\sigma|_N \in \hat{N}$, and that if λ is extended to $\mathfrak{j}^*$ by setting $\lambda(A_j) = 0$, $1 \leq j \leq r$, then there is a unique $t = (t_1, t_2, \dots, t_r) \in \mathbf{R}^r$ such that $\lambda + \sum_{j=1}^r t_j A_j^* \in O_\sigma$. For any j, $1 \leq j \leq r$, if $\nu = \sigma|_N$ is extended to $H_j = \exp(\mathbf{R}A_i + N)$ by formula (2), and χ_{t_i} is the character of H_j having differential $it_j(A_j^*|_{h_j})$, then $\sigma|_{H_j} = \chi_{t_i} \otimes \nu$.

LEMMA 4.6. *Let* $\alpha \in E$ *and let* $\lambda_n \in \Omega_\alpha$, $n = 0, 1, 2, \dots$, *such that the sequence* $\{\lambda_n\}_{n=1}^\infty$ *converges to* λ_0, *for each* $n \geq 0$, *let* $A_n \in R(\lambda_n, \mathfrak{g}) \sim \mathfrak{n}$, $\mathfrak{h}_n = \mathbf{R}A_n + \mathfrak{n}$, $H_n = \exp(\mathfrak{h}_n)$, *extend* λ_n *to* $\mathfrak{h}_n$ *by setting* $\lambda_n(A_n) = 0$, *define* $A_n^* \in \mathfrak{h}_n^*$ *by* $A_n^*(A_n) = 1$, $A_n^*|_{\mathfrak{n}} \equiv 0$, *let* $t_n \in \mathbf{R}$, *and let* $\rho_n = \eta_{H_n}^{-1}\mathrm{Ad}^*(H_n)(\lambda_n + t_n A_n^*)$. *Assume that* $A_n \to A_0$ *as* $n \to \infty$. *Then* $(\rho_n, H_n) \to (\rho_0, H_0)$ *if and only if* $t_n \to t_0$.

PROOF. We need only prove the "only if" part. Suppose that $(\rho_n, H_n) \to (\rho_0, H_0)$. Let π_0 an irreducible representation corresponding to λ_0 and let $\{\pi_n\}_{n=1}^\infty$ a sequence of representations corresponding to $\{\lambda_n\}_{n=1}^\infty$ as obtained in Theorem 3.5, so that $\pi_n \in \rho_n|_N$, $n \geq 0$. Extend π_n to H_n as in formula (2) so as to correspond to λ_n, and let χ_n be the character of H_n such that $\gamma_n = \chi_n \otimes \pi_n \in \rho_n$. Then by Corollary 4.2, there is $m > 0$ and $\{W_n\}_{n=0}^\infty \subset U(\mathfrak{g}_c)^{(m)}$ such that $W_n \to W_0$ and such that for each n, $W_n \in U((\mathfrak{h}_n)_c)$ and $\gamma_n(W_n) = t_n$. Now the general theory implies that $t_n \to t_0$. To see this, let $\Psi_0 \in C_c^\infty(G)$ and $v_0 \in H(\gamma_0)$ such that $\langle \gamma_0(\Psi_0)v_0, v_0\rangle \neq 0$. For each n, let Γ_n be the representation of $C_s^*(G)$ lifted from γ_n. Note that any $\Psi \in C_c^\infty(G)$ defines in a natural way an element $\tilde{\Psi}$ in $C_s^*(G)$ by setting $\tilde{\Psi}((K, x)) = \Psi(x)$, $K \in K(G)$, $x \in K$ such that for any $v \in H(\gamma_n)$, $\langle \Gamma_n(\tilde{\Psi})v, v\rangle = \langle \gamma_n(\Psi)v, v\rangle$. Now let $V_1, V_2, \dots, V_q \in U(\mathfrak{g}_c)$ such that for each n, $W_n = \sum_{j=1}^q a_j^{(n)} V_j$ with $a_j^{(n)} \in C$, $1 \leq j \leq q$, for each j, $a_j^{(0)} = \lim_n a_j^{(n)}$. Set $\Psi_j = V_j\Psi_0$, $1 \leq j \leq q$. Then by Lemma 2.2 of [4], there is, for each $n > 0$, $v_n \in H(\gamma_n)$ such that $\langle \Gamma_n(\tilde{\Psi}_j)v_n, v_n\rangle \to \langle \Gamma_0(\tilde{\Psi}_j)v_0, v_0\rangle$ as $n \to \infty$, $0 \leq j \leq q$. Thus $\langle \Gamma_n(W_n\Psi_0)v_n, v_n\rangle \to \langle \Gamma_0(W_0\Psi_0)v_0, v_0\rangle$ as $n \to \infty$, and we have

$$t_n = -\frac{\langle \gamma_n(W_n\Psi_0)v_n, v_n\rangle}{\langle \gamma_n(\Psi_0)v_n, v_n\rangle} \to -\frac{\langle \gamma_0(W_0\Psi_0)v_0, v_0\rangle}{\langle \gamma_0(\Psi_0)v_0, v_0\rangle} = t_0. \quad \text{Q.E.D.}$$

For each $\alpha \in E$, set $\tilde{U}_\alpha = \Theta^{-1}(\tilde{V}_\alpha)$.

THEOREM 4.7. *$\Theta|_{\tilde{U}_d}: \tilde{U}_\alpha \to \tilde{V}_\alpha$ is open, for each $\alpha \in E$.*

PROOF. Let $(\rho_0, H_0) \in \tilde{V}_\alpha$, and suppose that $\{(\rho_n, H_n)\}_{n=1}^{\infty}$ is a sequence in $\tilde{V}_\alpha$ which converges to (ρ_0, H_0). Let $\mathfrak{h}_n = \log(H_n)$, $n = 0, 1, 2, \ldots$, and let $l_0 \in \mathfrak{g}^*$ such that $l|_\mathfrak{h} \in O_{\rho_0}$. It is enough to show that there is a subsequence $\{(\rho_k, H_k)\}_{k=1}^{\infty}$ of $\{(\rho_n, H_n)\}_{n=1}^{\infty}$ and a corresponding sequence $\{l_k\}_{k=1}^{\infty}$ in $\mathfrak{g}^*$ such that $l_k|_{\mathfrak{h}_k} \in O_{\rho_k}$ for each k and $l_k \to l_0$. Note that we may assume that $\lambda_0 = l_0|_\mathfrak{n} \in \Omega_\alpha$. Let $\nu \in \hat{N}$ such that $\lambda_0 \in O_\nu$. By Lemma 4.3, there is $\nu_n \in \hat{N}$ such that $O_{\nu_n} \subset O_{\rho_n}|_\mathfrak{n}$, $n = 1, 2, 3, \ldots$, and such that the sequence $\{\nu_n\}_{n=1}^{\infty}$ converges to ν. Thus we have $\lambda_n \in O_{\nu_n} \subset O_{\rho_n}|_\mathfrak{n}$, $n = 1, 2, 3, \ldots$, such that $\{\lambda_n\}$ converges to λ_0. Now by restriction to a subsequence, we may assume that $\dim(\mathfrak{h}_n) = m$, $n = 0, 1, 2, \ldots$, and since Ω_α is open in $\bigcup_{\beta \geq \alpha} \Omega_\beta$ and $\{\lambda_n\} \in \bigcup_{\beta \geq \alpha} \Omega_\alpha$, we may assume that $\lambda_n \in \Omega_\alpha$ for all n. We proceed by induction on $\dim(\mathfrak{h}_n/\mathfrak{n}) = m - p$.

The case $m - p = 0$ is now trivial due to the above, so assume that $m > p$ and that the theorem is valid for sequences in $S_N(G)$ whose subgroup have dimension less than m. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a subsequence of $\{\lambda_n\}_{n=1}^{\infty}$ such that for some subalgebra $\mathfrak{j}_0$, the sequence $\mathfrak{j}_k = R(\lambda_k, \mathfrak{h}_k) + \mathfrak{n}$, $k = 1, 2, 3, \ldots$, converges to $\mathfrak{j}_0$. Let $J_k = \exp(\mathfrak{j}_k)$, $k \geq 0$. By Lemma 4.3, we have $\sigma_k \in \hat{J}_k$, $k = 0, 1, 2, \ldots$, such that $l_0|\mathfrak{j}_0 \in O_{\sigma_0}$, $O_{\sigma_k} \subset O_{\rho_k}|_{\mathfrak{j}_k}$, $k \geq 1$, and the sequence $\{(\sigma_k, J_k)\}_{k=1}^{\infty}$ converges to (σ_0, J_0). Suppose that $\dim J_0 < m$. By induction there is $l_k \in \mathfrak{g}^*$ such that $l_k|_{\mathfrak{j}_k} \in O_{\sigma_k}$, $k = 1, 2, 3, \ldots$, and such that the sequence $\{l_k\}_{k=1}^{\infty}$ converges to l_0. Now if $\rho_k: \mathfrak{h}_k^* \to \mathfrak{j}_k^*$ is the restriction mapping, then $\rho_k^{-1}(O_{\sigma_k}) \subset O_{\rho_k}$, $k \geq 1$ (cf. [**1**, Chapter II, §4.2]). Therefore $l_k|_{\mathfrak{h}_k} \in O_{\rho_k}$, and we are done. Hence by induction we have reduced to the case $\mathfrak{j}_k = \mathfrak{h}_k$, $k \geq 0$.

For each k, let $\{A_1^{(k)}, A_2^{(k)}, \ldots, A_r^{(k)}\} \subset R(\lambda_k, \mathfrak{h}_k)$ be a basis for $\mathfrak{h}_k$ mod $\mathfrak{n}$ such that for each $1 \leq j \leq r$, $A_j^{(0)} = \lim_k A_j^{(k)}$. Extending λ_k to $\mathfrak{h}_k$ by setting $\lambda_k(A_j^{(k)}) = 0$, $1 \leq j \leq r$, let $t_1^{(k)}, t_2^{(k)}, \ldots, t_r^{(k)}$ be real numbers such that $\lambda_k + \sum_{j=1}^{r} t_j^{(k)} A_k^{(k)^*} \in O_{\rho_k}$ (where $A_j^{(k)^*}$ is defined by $A_j^{(k)^*}(A_i^{(k)}) = \delta_{ij}$, $A_j^{(k)^*}|_\mathfrak{n} \equiv 0$). For each j, $i \leq j \leq r$, apply Lemma 4.6 to the sequence $\{(\rho_k|_{\exp(\mathbf{R}A_j^{(k)}+N)}, \exp(\mathbf{R}A_j^{(k)} + N))\}_{k=1}^{\infty}$ which converges to $(\rho_0|_{\exp(\mathbf{R}A_j^0+N)}, \exp(\mathbf{R}A_j^0 + N))$, and we obtain $t_j^{(0)} = \lim_k t_j^{(k)}$. Since $A_j^{(k)} \to A_j^{(0)}$, $1 \leq j \leq q$, it is clear that we may extend $\lambda_k + \sum_{j=1}^{q} t_j^{(k)} A_j^{(k)^*}$ to an element $l_k \in \mathfrak{g}^*$ such that the sequence $\{l_k\}_{k=1}^{\infty}$ converges to l_0. This finishes the proof. Q.E.D.

The following corollary is immediate.

COROLLARY 4.8. *$\eta_G|_{U_\alpha}: U_\alpha \to V_\alpha$ is a homeomorphism, for each α.*

Note that for each $\alpha \in E$, the dimensions of the orbits in U_α may vary, and the relative topology in U_α may not be T_1. Indeed, if N is abelian, then $E = \{\alpha_0\}$ and $U_{\alpha_0} = \mathfrak{g}^*$. Examples indicate that for each $\alpha \in E$, there is a finite partition of U_α, each element of which is T_2. Finally, the subsets U_α may be describable as Zariski-open subsets of algebraic varieties in $\mathfrak{g}^*$.

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